A Mathematical Basis for the Sugya in Bava Basra top of 88b

QUESTION: The Gemara tells us that a seller must give the buyer a bit more than an exact measure. The extra amount is called a "Hechra."

The Gemara explains that when an item is being measured with a balance scale, the amount of the Hechra is "one Tefach." This means, according to the Rishonim, that the seller does wait until the balance beam of the scale is perfectly level. Rather, he waits until the beam tips one Tefach in favor of the buyer.

How can a balance scale tip only a Tefach? If one side is heavier than the other, won't the heavier side continue to tip until it reaches the bottom?
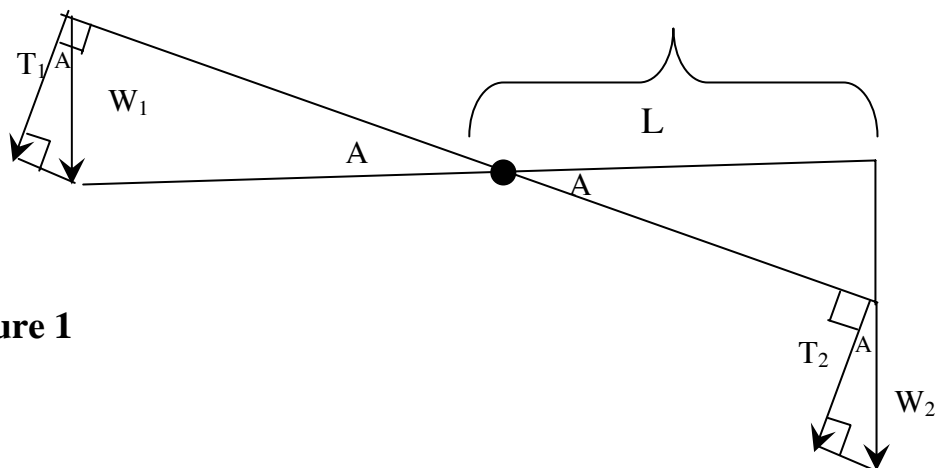
ANSWER:

Figure 1

Assume that the rod is straight and one weight (W_2) is heavier than the other (W_1). When the heavier weight causes one side of the scale to tip down (at an angle A), it will continue to exert more torque (rotational force) than the lighter weight does, until its side goes all the way to the bottom. It will not stop in the middle. The respective torques are:

$T_2 = W_2 (\cos A)$ and $T_1 = W_1 (\cos A)$.

Clearly, if $W_2 > W_1$, then $T_2 > T_1$ (until A reaches 90 degrees).

Figure 2
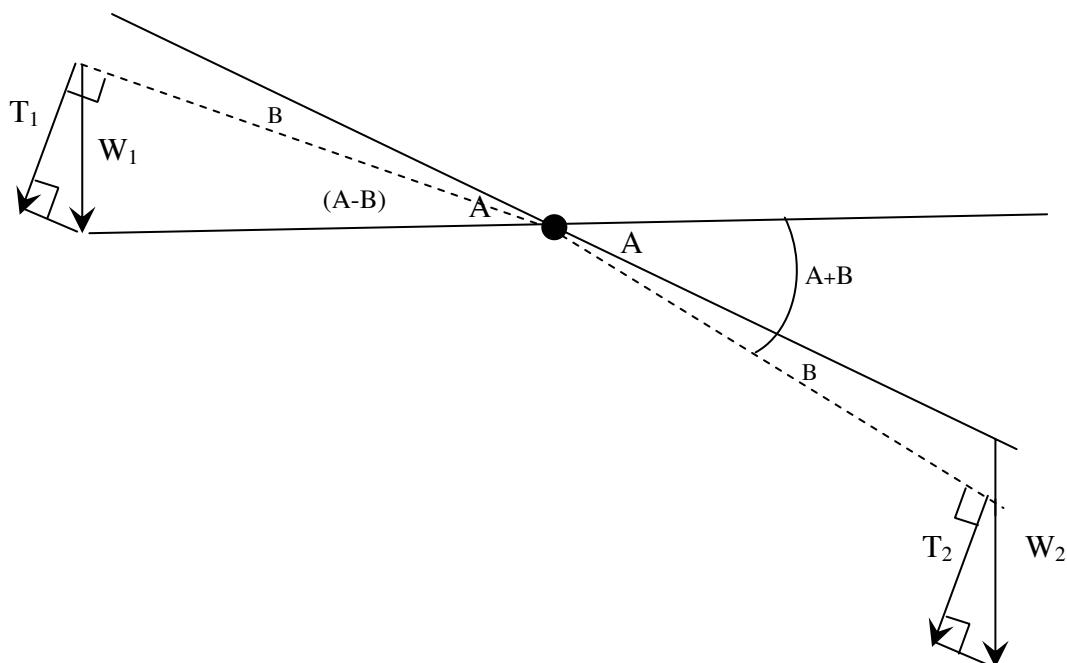
If we assume that the rod is warped at an angle of B, then both sides will be lower (the dashed lines). The angles of the sides are, respectively, (A-B) and (A+B), so we can calculate the torques with the equation:

$T_2 = W_2 * \cos(A+B)$ and $T_1 = W_1 * \cos(A-B)$.

We know that $\cos(A+B) = (\cos A)(\cos B) - (\sin A)(\sin B)$.
If B is small, $\cos(A+B) \approx (\cos A) - B(\sin A)$, if B is in radians. (1 degree $=\pi/180$ radians.)

Based on this and the above equation, we can conclude:
$T_2 \approx W_2((\cos A) - B(\sin A))$, and $T_1 \approx W_1((\cos A) + B(\sin A))$.

The torques T_2 and T_1 are equal when
$W_2 = W_1 *(\cos(A) + B\sin(A))/(\cos(A) - B\sin(A))$

The Me'iri and Yad Ramah explain that a Hechra of a Tefach is when the heavier pan is a Tefach lower than the lighter. If W_2 is half a Tefach below horizontal (the balanced position), and the length of the bar is four Tefachim (Rashbam 89a DH Yud-Beis), it follows that:

$\cos(A) \approx .98$, and $\sin(A) \approx .245$

The Rashbam (88b DH Echad) says that a Hechra of a Tefach is when the heavier weight is 1% more than the lighter, i.e.:

$1.01 = W_2/W_1 = 1 + (2B*(\sin A))/(\cos(A) - B(\sin A))$, so

$.01 = .49(B)/(.98 - .245B)$, i.e. $98 - .245B = 49B$, so

B (the warp angle) is about .02 radians $\approx$ 1.2 degrees.

When A=0, $T_2=W_2$ and $T_1=W_1$, the heavier side goes down. Clearly, the ratio $T_2/T_1 = (W_2/W_1)(\cos(A) + B(\sin A))/(\cos(A) - B(\sin A))$ keeps getting smaller when A increases from 0 until about 14 degrees. At that point, $\sin A \approx .245$, and the ratio becomes 1.

The Rashbam (89a DH Shalosh) implies that a Hechra of a Tefach is when the heavier pan is a Tefach below horizontal. If W_2 is a full Tefach below horizontal, then $(\sin A)=1/\sqrt{5} \approx .45$.

Since $1.01 = 1 + (2B(\sin A))/(\cos(A) - B(\sin A))$, we may conclude that $B \approx .9/90 = .01$ radians, or about .6 degrees.